Revisiting the "Rotating Spring Elongation" Problem (Variational Approach)

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Thanks to the letter from reader Awank-Newton, this article was corrected on 2013.01.30, mainly regarding the sign of the elastic potential energy. Previously, two consecutive errors were made, which coincidentally led to the correct answer. It has now been corrected. Refer to "Correction and Reflection on the Axiom of Equilibrium State".

In the following two articles, BoJone has introduced this "rotating spring elongation" problem and provided two methods of solution from two different perspectives. The former set up an integral equation and then transformed it into a differential equation to solve; the latter directly established a second-order differential equation from the perspective of elasticity. Both methods reach the same goal by different routes.
http://kexue.fm/archives/782/
http://kexue.fm/archives/826/

Today, after some further exploration into the calculus of variations, BoJone attempts to provide a new solution from a variational perspective (minimizing total energy). Similarly, let r be the distance from a point on the spring to the center of rotation after equilibrium is reached. Let the linear density at that point be $\lambda = \lambda(r)$, the mass of the spring from the center to that point be $m = m(r)$, the length before rotation be l_0, and the length after rotational equilibrium be l_1. Since the spring has reached an equilibrium state after rotation, by the Axiom of Equilibrium State (see the "Natural Extremum" series), equilibrium implies that the ~~total energy~~ "kinetic energy - potential energy" takes an extremal value.

The total kinetic energy of the spring is:

$$E_k = \int_0^{l_1} \frac{1}{2}[v(r)]^2 dm = \frac{1}{2}\int_0^{l_1} \lambda r^2 \omega^2 dr$$

Next, we find the total elastic potential energy of the spring. Since the spring is stretched, the elastic potential energy is negative [Note: as per the author's correction note]. The formula for the potential energy of an ideal spring is $E_p = \frac{1}{2}k(\Delta l)^2$, where Δl is the elongation. However, the spring after rotation is no longer an ideal spring (due to non-uniform elongation). The stiffness coefficient of the spring segment $[r, r+dr]$ is $\frac{M_0}{dm}k$, and the elongation is $(dr - \frac{dm}{\lambda_0})$ (where M_0 is the total mass of the spring and λ_0 is the original density, i.e., the average density). Thus, the total elastic potential energy is:

$$E_p = \frac{1}{2}\int_0^{l_1} \frac{M_0}{dm}k\left(dr - \frac{dm}{\lambda_0}\right)^2 = \frac{1}{2}\int_0^{l_1} M_0 k\left(\frac{1}{\lambda} + \frac{\lambda}{\lambda_0^2} - \frac{2}{\lambda_0}\right)dr$$

The ~~total energy~~ "kinetic energy - potential energy" (Lagrangian) is:

$$\begin{aligned} L &= \frac{1}{2}\int_0^{l_1} \left[\lambda r^2\omega^2 - M_0 k\left(\frac{1}{\lambda} + \frac{\lambda}{\lambda_0^2} - \frac{2}{\lambda_0}\right)\right] dr \\ &= \frac{1}{2}\int_0^{l_1} \left[\dot{m} r^2\omega^2 - M_0 k\left(\frac{1}{\dot{m}} + \frac{\dot{m}}{\lambda_0^2} - \frac{2}{\lambda_0}\right)\right] dr \end{aligned}$$

Now it becomes straightforward. Let $F = \dot{m}r^2\omega^2 - M_0 k\left(\frac{1}{\dot{m}} + \frac{\dot{m}}{\lambda_0^2} - \frac{2}{\lambda_0}\right)$. Setting its variation to zero, we have:

$$\frac{\partial F}{\partial m} - \frac{d}{dr}\left(\frac{\partial F}{\partial \dot{m}}\right) = 0$$

Since F does not explicitly contain m, we can obtain the first integral (see reference):

$$\frac{\partial F}{\partial \dot{m}} = C$$

That is:

$$r^2\omega^2 + \frac{M_0 k}{\dot{m}^2} - \frac{M_0 k}{\lambda_0^2} = C$$

When $r = l_1$, we should have $\dot{m} = \lambda_0$ (as there is no force stretching this segment at the free end, the density remains unchanged), which allows us to determine $C = l_1^2\omega^2$.

The subsequent processing is the same as in http://kexue.fm/archives/782/, so it will not be repeated here!

Through this article, we can catch a glimpse of the universality of the Axiom of Equilibrium State and the charm of using variational methods. We have bypassed force analysis and only needed to consider energy. The scalar nature of energy allows us to perform simple superposition calculations, which greatly reduces our cognitive load. Of course, this is just the tip of the iceberg. The calculus of variations shines in many fields with its unique charm!