

The Two Fixed-Center Problem

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When we study the motion of small celestial bodies near the Earth, if we treat the body and the Earth as a two-body system, it can at most be considered a zero-order approximation. If we use the circular restricted three-body problem model consisting of "Earth + Moon + small body," it can be considered a second-order approximation. So, what is the first-order approximation? BoJone believes it is the "two fixed-center problem" discussed in this article, also known as the "problem of two fixed centers" (two fixed-center problem). This is a special case of the restricted three-body problem. In this system, two primary bodies (or finite-mass bodies) remain fixed, and a third small body moves under the gravitational attraction of these two fixed primaries. Euler, Lagrange, Legendre, Jacobi, and others studied this problem very early on. Among them, Euler was the first to successfully find the integrals for this system. [Citation]

Additionally, the two fixed-center problem has another application. When studying the motion of artificial satellites, one might only consider Earth's gravity. However, since the Earth is not a perfect sphere, treating it as a single point mass is not entirely accurate. If it is split into two gravitational sources, the accuracy can be significantly improved. After all, the two fixed-center problem is completely integrable and can serve as a good intermediate orbit (somewhere between a conic section and an exact orbit).

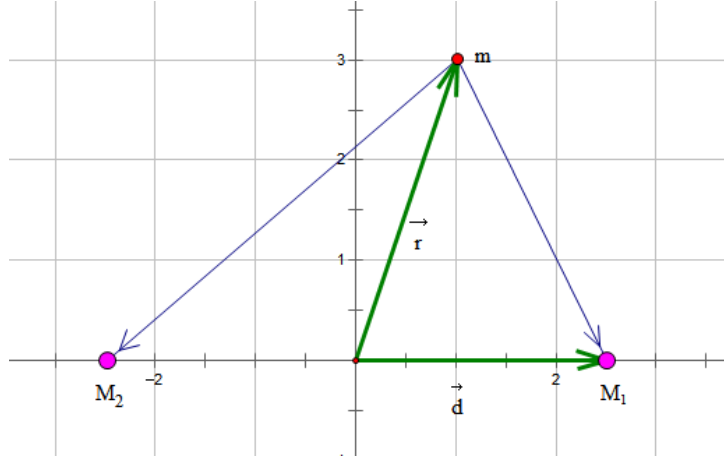


Figure 1: Two Fixed Centers - Force Analysis

With just a little knowledge of mechanical analysis, it is not difficult to list the equations of motion for the two fixed-center problem (assuming the mass of the moving object is 1):

$$\ddot{\vec{r}} = -GM_1 \frac{\vec{r} - \vec{d}}{|\vec{r} - \vec{d}|^3} - GM_2 \frac{\vec{r} + \vec{d}}{|\vec{r} + \vec{d}|^3} \quad (1)$$

Here, the origin is chosen at the center of the two fixed gravitational sources, and $\vec{d}$ is the position vector of body M_1 . This equation has an energy integral:

$$\frac{1}{2} \dot{\vec{r}}^2 - GM_1 \frac{1}{|\vec{r} - \vec{d}|} - GM_2 \frac{1}{|\vec{r} + \vec{d}|} = h \quad (2)$$

Next, we consider only the planar case, choosing appropriate units of length such that $\vec{d} = (1, 0)$, and denoting $GM_1 = \mu_1, GM_2 = \mu_2$. The kinetic energy of the system is $T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$, and the potential energy is $U = -\frac{\mu_1}{\sqrt{x^2+y^2+1-2x}} - \frac{\mu_2}{\sqrt{x^2+y^2+1+2x}}$. The conservation of energy integral is:

$$T + U = h \quad (3)$$

The Lagrangian is $L = T - U$. The key next step is to use an appropriate coordinate system to further simplify the Lagrange equations. **Euler brilliantly chose the elliptic coordinate system**, as seen in Wang Jiahe's *The Three-Body Problem*. However, we cannot know exactly how Euler arrived at this transformation. As a conjecture, BoJone provides a thought process. Since the potential energy U contains two square roots, we generally dislike dealing with such terms. Therefore, we should find a way to eliminate the two square roots through an appropriate coordinate transformation. The terms $(x^2 + y^2 + 1 - 2x)$ and $(x^2 + y^2 + 1 + 2x)$ inside the square roots differ only by a sign, which reminds us of the formula $(a \pm b)^2 = a^2 + b^2 \pm 2ab$. Thus, we let $x = 2ab$ and $a^2 + b^2 = x^2 + y^2 + 1$, which leads to:

$$y^2 = (a^2 - 1)(1 - b^2)$$

This further suggests the identities $\cos^2 \theta + \sin^2 \theta = 1$ and $\cosh^2 \varphi - 1 = \sinh^2 \varphi$. Consequently, we can consider:

$$\begin{aligned} x &= \cosh \varphi \cos \theta \\ y &= \sinh \varphi \sin \theta \end{aligned}$$

This is the desired coordinate transformation. However, **if we think this way, we cannot be certain that such a transformation will truly help the problem. Regardless, we should proceed, because even if it is wrong, it is worthwhile; it leads to deeper understanding or even unexpected discoveries.** The kinetic energy after transformation can be rewritten as:

$$T = \frac{1}{2}(\cosh^2 \varphi - \cos^2 \theta)(\dot{\varphi}^2 + \dot{\theta}^2)$$

The potential energy is:

$$U = -\frac{\mu_1}{\cosh \varphi - \cos \theta} - \frac{\mu_2}{\cosh \varphi + \cos \theta}$$

The energy integral is:

$$\begin{aligned} h &= T + U \\ &= \frac{1}{2}(\cosh^2 \varphi - \cos^2 \theta)(\dot{\varphi}^2 + \dot{\theta}^2) \\ &\quad - \frac{\mu_1}{\cosh \varphi - \cos \theta} - \frac{\mu_2}{\cosh \varphi + \cos \theta} \end{aligned} \quad (4)$$

The Lagrange equation is:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}} \right) = \frac{\partial L}{\partial \varphi}$$

Which is:

$$\frac{d}{dt}[(\cosh^2 \varphi - \cos^2 \theta)\dot{\varphi}] = \cosh \varphi \sinh \varphi(\dot{\varphi}^2 + \dot{\theta}^2) - \frac{\partial U}{\partial \varphi} \quad (5)$$

At this point, BoJone was stuck. However, **the genius Euler was able to brilliantly conceive this step**: multiplying each term by $(\cosh^2 \varphi - \cos^2 \theta)\dot{\varphi}$, we get:

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt}[(\cosh^2 \varphi - \cos^2 \theta)^2 \dot{\varphi}^2] \\ &= (\cosh^2 \varphi - \cos^2 \theta)(\dot{\varphi}^2 + \dot{\theta}^2) \cosh \varphi \sinh \varphi \dot{\varphi} \\ &\quad - \frac{\partial U}{\partial \varphi}(\cosh^2 \varphi - \cos^2 \theta)\dot{\varphi} \end{aligned} \quad (6)$$

Noticing that the right side contains $(\cosh^2 \varphi - \cos^2 \theta)(\dot{\varphi}^2 + \dot{\theta}^2) = 2T$, it can be written as:

$$\frac{1}{2} \frac{d}{dt} [(\cosh^2 \varphi - \cos^2 \theta)^2 \dot{\varphi}^2] = 2(h - U) \cosh \varphi \sinh \varphi \dot{\varphi} + \frac{\partial(h - U)}{\partial \varphi} (\cosh^2 \varphi - \cos^2 \theta) \dot{\varphi} \quad (7)$$

The right side is:

$$\dot{\varphi} \frac{\partial}{\partial \varphi} [(h - U)(\cosh^2 \varphi - \cos^2 \theta)] \quad (8)$$

Substituting the expression for U , equation (8) can be further simplified:

$$\dot{\varphi} \frac{\partial}{\partial \varphi} [h(\cosh^2 \varphi - \cos^2 \theta) + \mu_1(\cosh \varphi + \cos \theta) + \mu_2(\cosh \varphi - \cos \theta)] \quad (9)$$

Since φ and θ are independent, and the θ and φ parts are already separated, the partial derivative of the θ part with respect to φ is 0. Therefore, equation (8) is equivalent to:

$$\dot{\varphi} \frac{\partial}{\partial \varphi} [h \cosh^2 \varphi + (\mu_1 + \mu_2) \cosh \varphi] = \frac{d}{dt} [h \cosh^2 \varphi + (\mu_1 + \mu_2) \cosh \varphi] \quad (10)$$

Thus, equation (6) becomes:

$$\frac{1}{2} \frac{d}{dt} [(\cosh^2 \varphi - \cos^2 \theta)^2 \dot{\varphi}^2] = \frac{d}{dt} [h \cosh^2 \varphi + (\mu_1 + \mu_2) \cosh \varphi] \quad (11)$$

Now we can integrate both sides:

$$\frac{1}{2} (\cosh^2 \varphi - \cos^2 \theta)^2 \dot{\varphi}^2 = h \cosh^2 \varphi + (\mu_1 + \mu_2) \cosh \varphi + C \quad (12)$$

From the energy integral (4), we can transform it to get:

$$\frac{1}{2} (\cosh^2 \varphi - \cos^2 \theta)^2 (\dot{\varphi}^2 + \dot{\theta}^2) = h(\cosh^2 \varphi - \cos^2 \theta) + (\mu_1 + \mu_2) \cosh \varphi + (\mu_1 - \mu_2) \cos \theta \quad (13)$$

Comparing this with (12), we can separate the variables to obtain:

$$\frac{1}{2} (\cosh^2 \varphi - \cos^2 \theta)^2 \dot{\theta}^2 = -h \cos^2 \theta + (\mu_1 - \mu_2) \cos \theta - C \quad (14)$$

Dividing (14) by (12) yields:

$$\left(\frac{d\theta}{d\varphi} \right)^2 = \frac{-h \cos^2 \theta + (\mu_1 - \mu_2) \cos \theta - C}{h \cosh^2 \varphi + (\mu_1 + \mu_2) \cosh \varphi + C} \quad (15)$$

We quickly obtain:

$$\int \frac{d\theta}{\sqrt{-h \cos^2 \theta + (\mu_1 - \mu_2) \cos \theta - C}} = \int \frac{d\varphi}{\sqrt{h \cosh^2 \varphi + (\mu_1 + \mu_2) \cosh \varphi + C}} \quad (16)$$

Both sides are in integrable form, and both are **elliptic integrals**. In fact, from the perspective of complex numbers, it is easy to see that the integrals on both sides are actually equivalent. Since $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\cos x = \frac{e^{ix} + e^{-ix}}{2}$, the two differ only by an imaginary unit i , and their integrals are easily transformed into each other.

This concludes the two fixed-center problem. Since we are only seeking the solution process and do not require practical application testing (which is usually handled by computers), there is no need to write out the full forms of the integrals. In fact, it is not necessary, as expressing the solution of a differential equation in integral form is sufficient. For more content, one can refer to Wang Jiahe's *The Three-Body Problem*.

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