

The “Two-Body + Constant Force” Problem

Su Jianlin

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After looking at the “Two Fixed Centers” problem, we might as well look at a seemingly simpler mechanics problem: based on the gravitational attraction of a fixed mass, we add a constant force and study the motion of a small body in such a force field.

At first glance, this problem appears much simpler than the “Two Fixed Centers” problem; at least the equations of motion look simpler:

$$\ddot{\vec{r}} = -GM \frac{\vec{r}}{|\vec{r}|^3} + \vec{F} \quad (1)$$

where $\vec{F}$ is a constant vector. However, what is quite surprising is that this problem is **essentially the same as the “Two Fixed Centers” problem**; it can be regarded as a limiting case of the Two Fixed Centers problem. Furthermore, their solution methods are strikingly similar. Let us analyze this process below.

First, it is easy to write the energy conservation integral for this system:

$$1/2 \dot{\vec{r}}^2 - GM \frac{1}{|\vec{r}|} - \vec{F} \cdot \vec{r} = h \quad (2)$$

Similarly, we only consider the planar case. By choosing an appropriate reference frame such that $\vec{F} = (F, 0)$, we can denote the kinetic energy of the system as $T = 1/2(\dot{x}^2 + \dot{y}^2)$ and the potential energy as $U = -\frac{GM}{\sqrt{x^2+y^2}} - Fx$. The energy is

$$T + U = h \quad (3)$$

The Lagrangian is $L = T - U$.

To eliminate the square root in the potential function U , the first thing that comes to mind is using polar coordinates. Indeed, this coordinate system can eliminate the square root, but it does not allow us to proceed further. So, is there any other way to eliminate the square root?

A faithful fan of mathematics would know that over 2000 years ago, the great Pythagoras came up with an expression for “Pythagorean triples”: $(a^2 - 1)^2 + (2a)^2 = (a^2 + 1)^2$. Later, people further discovered the identity $(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2$. We find that the form on the left side corresponds exactly to $x^2 + y^2$, so we can consider using this formula for a transformation.

Now the question arises: in $x^2 + y^2$, x and y are symmetric, but in $(a^2 - b^2)^2 + (2ab)^2$, the two quantities inside the parentheses are not symmetric. Thus, we need to determine whether $x = a^2 - b^2$ or $x = 2ab$ is more suitable. The correct answer is:

$$\begin{aligned} x &= a^2 - b^2 \\ y &= 2ab \end{aligned} \quad (4)$$

Why choose this? BoJone did not have a good way to judge at first; initially, BoJone swapped the two, but the calculation could not be simplified at the end, so they were swapped back. Therefore, this is a type of “ex post facto judgment method.”

After the transformation: $T = 2(a^2 + b^2)(\dot{a}^2 + \dot{b}^2)$, $U = -\frac{GM}{a^2 + b^2} - F(a^2 - b^2)$.

Now this transformation looks quite good; it **maintains the original symmetry of the equations**. The subsequent processing method is almost identical to the “Two Fixed Centers” problem. Please see:

The Lagrange equation is $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{a}} \right) = \frac{\partial L}{\partial a}$.

That is, $\frac{d}{dt}[4\dot{a}(a^2 + b^2)] = 4a(\dot{a}^2 + \dot{b}^2) - \frac{\partial U}{\partial a}$.

Multiplying each term by $\dot{a}(a^2 + b^2)$ gives:

$$\frac{d}{dt}[2\dot{a}^2(a^2 + b^2)^2] = 4a\dot{a}(\dot{a}^2 + \dot{b}^2)(a^2 + b^2) - \frac{\partial U}{\partial a}\dot{a}(a^2 + b^2) \quad (5)$$

Substituting using the energy conservation integral:

$$\frac{d}{dt}[2\dot{a}^2(a^2 + b^2)^2] = 2a\dot{a}(h - U) - \frac{\partial U}{\partial a}\dot{a}(a^2 + b^2) \quad (6)$$

The right side becomes:

$$\begin{aligned} & \dot{a} \frac{\partial}{\partial a} [(h - U)(a^2 + b^2)] \\ &= \dot{a} \frac{\partial}{\partial a} [h(a^2 + b^2) + GM + F(a^4 - b^4)] \\ &= \dot{a} \frac{\partial}{\partial a} (ha^2 + Fa^4) \\ &= \frac{d}{dt} (ha^2 + Fa^4) \end{aligned} \quad (7)$$

At this point, we should understand why we chose $x = a^2 - b^2$ instead of $x = 2ab$. If the latter were chosen, it would be impossible to separate the terms of a and b from each other.

Thus, we have the integral: $2\dot{a}^2(a^2 + b^2)^2 = ha^2 + Fa^4 + C$.

From the transformation of the energy integral: $2(\dot{a}^2 + \dot{b}^2)(a^2 + b^2)^2 = h(a^2 + b^2) + GM + F(a^4 - b^4)$.

Comparing the two equations, we have $2\dot{b}^2(a^2 + b^2)^2 = hb^2 + GM - Fb^4 - C$.

Which leads to:

$$\left(\frac{db}{da} \right)^2 = \frac{hb^2 - Fb^4 + (GM - C)}{ha^2 + Fa^4 + C} \quad (8)$$

Integrating gives:

$$\int \frac{da}{\sqrt{ha^2 + Fa^4 + C}} = \int \frac{db}{\sqrt{hb^2 - Fb^4 + (GM - C)}} \quad (9)$$

See that? The entire process is a complete “**replica**” of the solution process for the “Two Fixed Centers” problem!!

In fact, the coordinate transformation $x = a^2 - b^2, y = 2ab$ is known in mathematics as the **parabolic coordinate transformation** ^[Citation]. Does this remind us of knowledge related to conic sections? A parabola is actually a limiting case of an ellipse (where the eccentricity tends to 1), so parabolic coordinates are also a limiting case of elliptical coordinates, and this problem is a limiting case of the “Two Fixed Centers” problem. As for how they correspond exactly, BoJone has not yet clarified the details, so I will not say more.

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