A Problem on Comparing Function Values

Jianlin Su

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A few days ago, in the mathematics paper of the Yunfu Grade 11 final exam, there was a problem that left a deep impression on me. At that time, I was unable to prove it formally and only arrived at the answer by using specific examples. I have just given it some more thought and will present the proof process here. The problem is as follows:

> A function $f(x)$ defined on $(0, +\infty)$ satisfies $xf'(x) \leq f(x)$. For any $0 < a < b$, compare the magnitudes of $af(b)$ and $bf(a)$.

Since this was a fill-in-the-blank question, I obtained the result by testing two examples:

- Let $f(x) = x$, then $af(b) = ab$ and $bf(a) = ba$, so $af(b) = bf(a)$.
- Let $f(x) = x + 1$, then $af(b) = a(b + 1) = ab + a$ and $bf(a) = b(a + 1) = ba + b$. Since $a < b$, it follows that $af(b) < bf(a)$.

Thus, the answer is $af(b) \leq bf(a)$.

Of course, what readers want to know is not just the answer, but how to come up with these examples. During the exam, BoJone was momentarily without a clear direction, so I thought about replacing the inequality $xf'(x) \leq f(x)$ with an equality. Solving the differential equation $xf'(x) = f(x)$ yields $f(x) = kx$, where k is an arbitrary constant. Substituting this into $af(b)$ and $bf(a)$ results in an identity. By slightly modifying this example (adding a constant term), I obtained the inequality case. Thus, the problem was solved.

However, truly good mathematics requires rigorous proof. This is what I came up with this evening:

To compare the magnitudes of $af(b)$ and $bf(a)$, we only need to compare the magnitudes of $\frac{af(b)}{ab}$ and $\frac{bf(a)}{ab}$, which is equivalent to comparing $\frac{f(b)}{b}$ and $\frac{f(a)}{a}$. This effectively means proving the monotonicity of the function $\frac{f(x)}{x}$.

The derivative is:

$$\left(\frac{f(x)}{x}\right)' = \frac{xf'(x) - f(x)}{x^2}$$

The problem states that $xf'(x) \leq f(x)$, therefore:

$$\left(\frac{f(x)}{x}\right)' = \frac{xf'(x) - f(x)}{x^2} \leq 0$$

This implies that $\frac{f(x)}{x}$ is either a decreasing function or a constant function. Given that $0 < a < b$, we have:

$$\frac{f(b)}{b} \leq \frac{f(a)}{a}$$

which simplifies to:

$$af(b) \leq bf(a)$$

In fact, BoJone believes that this problem should not have been a fill-in-the-blank question but rather a comprehensive problem requiring the full solution process. Such problems truly qualify as "gymnastics for exercising the mind."