# Spring Two-Body Motion

Jianlin Su

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This was a question from our final exam, one of the physics problems in the Science Comprehensive section.

> A massless ideal spring has a particle of mass $m$ attached to each end (A on the left, B on the right). A is given an initial velocity $v_0$ to the right, causing the system to begin moving. What is the elastic potential energy when the spring is compressed to its minimum length? And what is the maximum velocity of particle B?

High school students solve this by combining the conservation of momentum and the conservation of energy. However, I hope to use differential equations to grasp the overall information of this motion and, in passing, verify whether the spring can completely transfer the velocity $v_0$ of A to B.

First, we choose the right as the positive direction. Since both A and B move along the same straight line, this is a one-dimensional motion problem. Let the coordinate of A be $x$, the coordinate of B be $y$, the natural length of the spring be $l_0$, and the spring constant be $k$. Clearly, the extension of the spring is $\Delta l = y - x - l_0$. According to Hooke's Law, the elastic force acting on A is $F = k(y - x - l_0)$. Therefore, we can set up the following system of differential equations:

$$m\ddot{x} = F \tag{1}$$

$$m\ddot{y} = -F \tag{2}$$

$$F = k(y - x - l_0) \tag{3}$$

The initial conditions are: At $t = 0$, $x = 0$, $y = l_0$, $v_A = \dot{x} = v_0$, $v_B = \dot{y} = 0$.

Adding (1) and (2) gives: $\ddot{x} + \ddot{y} = 0$. Integrating this yields:

$$\begin{aligned} \dot{x} + \dot{y} &= v_0 \\ x + y &= v_0 t + l_0 \end{aligned} \tag{4}$$

(Here, the integration constants have been determined according to the initial conditions.)

Subtracting (1) from (2) gives:

$$m(y - x)'' = -2F$$

Or written as:

$$(y - x - l_0)'' = -\frac{2k}{m}(y - x - l_0) \tag{5}$$

Let $y - x - l_0 = z$, then (5) becomes:

$$z'' = -\frac{2k}{m} z$$

This is a second-order linear differential equation with constant coefficients. The characteristic roots are $\pm i\sqrt{\frac{2k}{m}}$, so the general solution can be written as:

$$z = C_1 \sin\left(\sqrt{\frac{2k}{m}}t\right) + C_2 \cos\left(\sqrt{\frac{2k}{m}}t\right) = y - x - l_0 \tag{6}$$

When $t = 0$, it is clear that $z = 0$, so $C_2 = 0$. Differentiating $z$ with respect to time gives:

$$\dot{z} = C_1\sqrt{\frac{2k}{m}} \cos\left(\sqrt{\frac{2k}{m}}t\right) = \dot{y} - \dot{x}$$

When $t = 0$, it is clear that $\dot{z} = -v_0$, so $C_1 = -v_0\sqrt{\frac{m}{2k}}$.

Thus:

$$y - x = l_0 - v_0\sqrt{\frac{m}{2k}} \sin\left(\sqrt{\frac{2k}{m}}t\right) \tag{7}$$

Combining (4) and (7), we can solve for $x$ and $y$:

$$x = \frac{v_0 t}{2} + \frac{v_0}{2}\sqrt{\frac{m}{2k}} \sin\left(\sqrt{\frac{2k}{m}}t\right)$$
$$y = l_0 + \frac{v_0 t}{2} - \frac{v_0}{2}\sqrt{\frac{m}{2k}} \sin\left(\sqrt{\frac{2k}{m}}t\right)$$

Now everything has an answer ^_^. It turns out that B really can attain the velocity $v_0$.

**When encountering problems, analyzing them with the knowledge you have learned can broaden your horizons and open up your thinking ^_^!**