

The Grain Collection Problem (1)

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In rural areas, July is a busy month. Farmers are occupied with harvesting rice, drying it after the harvest, and simultaneously preparing for "seedling transplantation" for the next round of cultivation. BoJone's family is naturally no exception. However, our family has relatively little farmland (about 1 mu, approx. 0.16 acres), so the harvester finishes the job in a few minutes, and the grain is dried in just two or three days. Nevertheless, while drying the grain, BoJone considered a "grain collection" problem:

When drying grain, one must first spread the piles of grain out, laying them flat in a thin layer on the ground. In the evening or when rain is impending (which is the worst-case scenario, as the grain might get drenched), it must be collected. The problem originates here: generally, we spread the grain uniformly into a rectangle. To which point, either inside or outside the rectangle, should all the grain be pushed to minimize the work done?

This problem can be further generalized. For instance, for grain spread in an arbitrary shape (such as a triangle), to which point should it be concentrated to make the task most "effortless"? For a fixed mass of grain, into what shape should it be spread to make the collection most "effortless"? Of course, the solution to this problem is not only applicable to "grain collection" but can also be applied to many planning and construction scenarios. For example, if a service center is to be built in a city with a roughly uniform population distribution, where should this service center be located? This is somewhat similar to the Fermat point problem we discussed previously; however, the Fermat point only considers the distance to three points, whereas this problem must consider the distance to all points.

Without further ado, let us look at the first question: To which point, inside or outside the rectangle, should all the grain be pushed to minimize the work done? In fact, by symmetry, we can easily prove that this point is the geometric center of the rectangle (perhaps the reader would like to try proving this?). However, BoJone is not satisfied with qualitative analysis; I wish to perform a quantitative calculation.

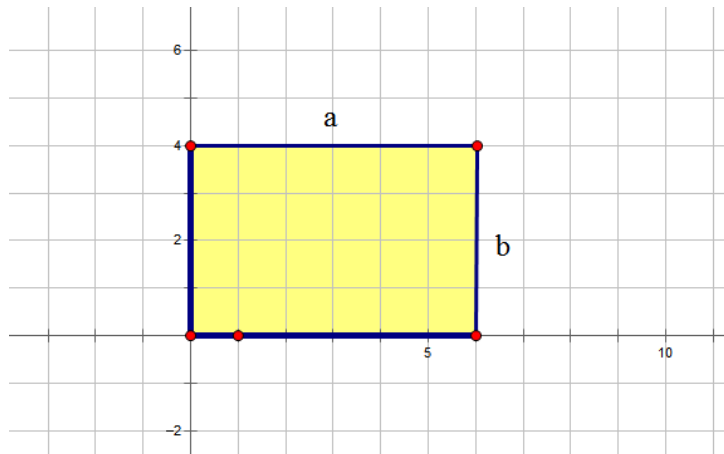


Figure 1: Rectangular Grain Field

Let us first consider the case where all the grain is piled into one corner (the origin). Assume the surface density of the grain is 1. The work done to collect the grain should be proportional to both mass and distance. Thus, the work done to move a small rectangular element of grain with side lengths dx and dy back to the origin is $\sqrt{x^2 + y^2}dxdy$. The work done to move the entire rectangular field back to the origin should be the double integral:

$$W = \iint \sqrt{x^2 + y^2}dxdy$$

(The integration region is a rectangle with side lengths a and b).

For friends who have not yet encountered double integrals, here is a brief introduction. The calculation of a double integral can be transformed into an iterated integral. That is, first treat y as a known constant and perform a definite integral with respect to x . After that calculation, treat y as the variable and perform a definite integral with respect to y . The integration region of a single integral (the definite integral we usually encounter) is a line (for example, the integration region of $\int_a^b f(x)dx$ is a line segment from a to b on the x -axis), whereas the integration region of a double integral is a plane. Therefore, double integrals are much more complex than single integrals. For instance, in the step "first treat y as a known constant and integrate with respect to x ," one must write the expression for x in terms of y for the integration region; that is, the difference between the upper and lower limits of the definite integral represents x as a function of y .

The double integral W can be calculated directly, but for simplification, I hope to transform it using polar coordinates:

$$W = \iint r^2 dr d\theta$$

Does this look much simpler? We treat θ as known and integrate with respect to r . But how do we write the limits of integration here? As seen in the figure, in the lower-right right-angled triangle region, we have $r = \frac{a}{\cos\theta}$ (which is the equation of the rightmost side of the rectangle). Thus, the integration interval can be written as $[0, \frac{a}{\cos\theta}]$, i.e.,

$$W_1 = \int \left(\int_0^{a/\cos\theta} r^2 dr \right) d\theta = \int \left(\frac{1}{3} r^3 \Big|_0^{a/\cos\theta} \right) d\theta = \int \frac{a^3}{3 \cos^3 \theta} d\theta$$

According to the derivation here, we have:

$$W_1 = \frac{a^3}{4} \left(\frac{2 \sin \theta}{\cos^2 \theta} + \ln \left| \frac{1 + \sin \theta}{1 - \sin \theta} \right| \right)$$

where $\theta = \arctan(b/a)$.

However, this is only half of it. The integral for the upper-left side is equal to:

$$\begin{aligned} W_2 &= \int \left(\int_0^{b/\cos\theta} r^2 dr \right) d\theta = \int \left(\frac{1}{3} r^3 \Big|_0^{b/\cos\theta} \right) d\theta = \int \frac{b^3}{3 \cos^3 \theta} d\theta \\ &= \frac{b^3}{4} \left(\frac{2 \sin \theta}{\cos^2 \theta} + \ln \left| \frac{1 + \sin \theta}{1 - \sin \theta} \right| \right) \end{aligned}$$

where $\theta = \arctan(a/b)$.

Is the result already making our heads spin? Yes, BoJone's head is spinning too, but let us persevere. W can be written as:

$$\begin{aligned} &\frac{a^3}{4} \left(\frac{2b\sqrt{a^2 + b^2}}{a^2} + 2 \ln \left| \frac{\sqrt{a^2 + b^2} + b}{a} \right| \right) + \frac{b^3}{4} \left(\frac{2a\sqrt{a^2 + b^2}}{b^2} + 2 \ln \left| \frac{\sqrt{a^2 + b^2} + a}{b} \right| \right) \\ &= ab\sqrt{a^2 + b^2} + \frac{1}{2}a^3 \ln \left| \frac{\sqrt{a^2 + b^2} + b}{a} \right| + \frac{1}{2}b^3 \ln \left| \frac{\sqrt{a^2 + b^2} + a}{b} \right| \end{aligned}$$

We have managed to calculate it, but such a cumbersome formula makes us hesitate and seek an alternative approach... (To be continued)

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