

[Euler Mathematics] Finding a Rigorous Answer

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September 9, 2013

In previous articles, we have discussed Euler mathematics. Generally speaking, Euler mathematics refers to creative and intuitive techniques and methods that can derive beautiful results, even if the methods themselves are not strictly rigorous. However, in many cases, rigor and intuition are only a step away. What follows is a problem from my final exam in "Mathematical Analysis" last semester, and the inspiration for my solution came from "Euler Mathematics."

Let $\{a_n\}$ be an increasing sequence of positive numbers. Prove that the convergence of $\sum_{n=1}^{\infty} \left(1 - \frac{a_n}{a_{n+1}}\right)$ is equivalent to the convergence of $\{a_n\}$.

It is said that the reference answer used the Cauchy convergence criterion for sequences. I didn't look at it closely; instead, I explored my own more intuitive method. This is the embodiment of "I do not understand what I can not create." Below is my line of thought.

Thinking

A powerful method for determining whether a series converges is the **Integral Test**, which I usually think of first. Although no specific function is given here and integration cannot be performed directly, we can still gain some inspiration.

Let $a_n \equiv a(n)$. In cases where $a(n)$ changes smoothly, the following approximations are quite good: $a'(x) \sim a(x+1) - a(x)$ and $a(x) \sim a(x+1)$. Therefore, $1 - \frac{a(x)}{a(x+1)} \sim \frac{a'(x)}{a(x)}$. According to the Integral Test, we have:

$$S = \sum_{n=1}^{\infty} \left(1 - \frac{a_n}{a_{n+1}}\right) \sim \int_1^{+\infty} \frac{a'(x)}{a(x)} dx = \ln a(x) \Big|_1^{+\infty}$$

It is worth noting that every step here is non-rigorous and might not even hold, but it allows us to understand the problem from a different perspective. Methodologically, each step is representative, such as **approximating differences with derivatives and sums with integrals**. The final result also hints at the connection to the problem: if $a(\infty)$ exists, the integral exists, and the series converges.

Setting aside rigor for a moment, the most important thing this thinking brings us is: $\ln a(x)$!

Sufficiency

This implies $S_n \sim \ln[a(n)]$. Thus, we can consider:

$$\ln a_{n+1} - \ln a_n = -\ln \frac{a_n}{a_{n+1}} = -\ln \left[1 - \left(1 - \frac{a_n}{a_{n+1}}\right)\right]$$

Using the expansion $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$, we have $-\ln(1-x) > x$. Thus, we can write:

$$\ln a_{n+1} - \ln a_n > 1 - \frac{a_n}{a_{n+1}}$$

Using this inequality, we obtain:

$$\ln a_{n+1} - \ln a_1 > \sum_{i=1}^n \left(1 - \frac{a_i}{a_{i+1}}\right)$$

This proves that the convergence of $\{a_n\}$ implies the convergence of the series, which establishes **sufficiency**.

(Before reaching the result, we do not know whether we will prove sufficiency or necessity. It only becomes clear after analysis. Regardless, as long as we have such an inequality, we can prove one of them. This is part of our confidence. Next, we just need to construct the opposite inequality.)

Necessity

To prove **necessity**, we need to find a similar inequality but with the opposite sign. From $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$, we see that no matter where we truncate, we won't get an inequality with the opposite sign. So we consider $-\ln(1-x) < 2x$. This is not always true, but it holds at least for $x \in [0, \frac{1}{2}]$. Under the premise that $\sum_{n=1}^{\infty} \left(1 - \frac{a_n}{a_{n+1}}\right)$ converges, only a finite number of terms of $1 - \frac{a_n}{a_{n+1}}$ can be greater than $\frac{1}{2}$; otherwise, there would be a contradiction. Thus, there exists an N such that for $n > N$, $1 - \frac{a_n}{a_{n+1}} \in (0, \frac{1}{2}]$, which ensures the applicability of $-\ln(1-x) < 2x$. That is, for sufficiently large n :

$$\ln a_{n+1} - \ln a_n < 2 \left(1 - \frac{a_n}{a_{n+1}}\right)$$

Then we can write:

$$\ln a_{n+1} - \ln a_{N+1} < 2 \sum_{i=N+1}^n \left(1 - \frac{a_i}{a_{i+1}}\right)$$

Thus, the convergence of the latter implies the convergence of the former, which proves **necessity**.

Summary

In any case, thinking in multiple directions is always beneficial. It is great to be able to think of the same method as the reference answer, but we should not be limited to specific methods. Instead, we should center on ourselves and find a system of thinking that belongs to us; only then can we be said to have "created" the answer. Additionally, regarding Euler mathematics, teachers won't care what you write on your scratch paper, so during calculations, brainstorm as much as possible and associate various methods. As the saying goes, "Stones from other hills may serve to polish the jade of this one." It is worth noting that this process is not just for exams but should run through our entire learning process. In this regard, the role of "hindsight" is quite significant!

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