When Probability Meets Complex Variables: Random Walk and Path Integral

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In our previous article, we saw that the probability distribution of a random walk is normal. In probability theory, we learn that the normal distribution is (almost) the most important type of distribution. Both the random walk model and the normal distribution are widely applied. We might ponder a question: does the random walk create the normal distribution, or does the normal distribution create the random walk? In other words, which one is more fundamental? Based on what I have read so far, the random walk seems more fundamental. The random walk corresponds exactly to the ubiquitous random uncertainty (such as measurement errors), and its distribution happens to be the normal distribution. This is why the normal distribution is so widely applied—because random uncertainty is everywhere.

Next, let us consider another way to describe the random walk. In principle, it is more extensive and profound, known by the famous name: "Path Integral."

Introduction to Path Integral

When talking about path integrals, one cannot avoid mentioning Feynman. The pioneer of the path integral was the genius physicist I highly admire, Richard Feynman. At that time, the path integral emerged as a third equivalent description of quantum mechanics. Compared to Heisenberg's matrix mechanics and Schrödinger's wave mechanics, Feynman's path integral is mathematically the most cumbersome, but its underlying idea is the easiest to understand and possesses a universality that has led it to become the mainstream formulation of modern quantum field theory. This further demonstrates that mathematical complexity is not where the true difficulty of a problem lies. If the ideas are simple and the logic is clear, even the most tedious calculations become straightforward. In the words of physicists: without mathematical details, we can still accomplish many things; but without physical insight, we can do nothing.

The idea of the path integral is truly simple. Usually, we consider probability density—that is, the probability of an event occurring within a small region. The path integral considers the probability (or probability amplitude in quantum mechanics) of an event occurring along a specific path. Then, by employing a certain integral measure, it sums the probabilities of all possible paths to obtain the total probability.

While the concept of the path integral is simple, handling it clearly in mathematics requires a lot of work, which *Scientific Space* will introduce in future specialized topics. Until then, Feynman's *Quantum Mechanics and Path Integrals* is an excellent reference book (perhaps the best one).

Path Integral of the Random Walk

According to the results from the previous article, we know that in a random walk starting from x_0, the probability density of reaching x_n after time t is:

$$\frac{1}{\sqrt{2\pi\alpha t}}\exp\left(-\frac{(x_n-x_0)^2}{2\alpha t}\right) \tag{1}$$

We divide the time into n equal segments, each with a length of $\Delta t=\frac{t}{n}$. At time $i\Delta t$, the position of the particle is x_i. The probability density for the particle to move from x_i to x_{i+1} is:

$$\frac{1}{\sqrt{2\pi\alpha\Delta t}}\exp\left(-\frac{(x_{i+1}-x_i)^2}{2\alpha\Delta t}\right) \tag{2}$$

Therefore, the probability density for the particle to pass through $x_1,x_2,\dots,x_{n-1},x_n$ in sequence is:

$$\left(\frac{1}{\sqrt{2\pi\alpha\Delta t}}\right)^n\exp\left(-\frac{(x_1-x_0)^2+(x_2-x_1)^2+\dots+(x_n-x_{n-1})^2}{2\alpha\Delta t}\right) \tag{3}$$

Omitting the preceding factors and taking the limit $\Delta t\to 0$, we find that the probability of the particle following the path $x=x(t)$ is proportional to:

$$\exp\left(-\int\frac{\dot{x}^2}{2\alpha}dt\right) \tag{4}$$

This gives us the probability of a specific path. Using the path integral method from quantum mechanics, we can start from this and conversely derive the probability density:

$$\frac{1}{\sqrt{2\pi\alpha t}}\exp\left(-\frac{(x_n-x_0)^2}{2\alpha t}\right)=\int\exp\left(-\int\frac{\dot{x}^2}{2\alpha}dt\right)\mathcal{D}x(t) \tag{5}$$

This is the path integral concept applied to the random walk. As for the specific mathematical details, we will discuss them in the future.