

[Understanding Riemannian Geometry] 3. Geodesics

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Geodesics

Riemannian metrics should not be difficult to understand. In differential geometry textbooks, we have already studied the "first fundamental form" of a surface. In fact, the two are the same thing, just viewed from different perspectives. Differential geometry treats a surface as a two-dimensional subset of three-dimensional space, whereas Riemannian geometry studies geometric problems intrinsically from the two-dimensional surface itself.

What questions does geometry concern itself with? In fact, **geometry is concerned with “objective entities” that are independent of transformations (or things that remain invariant under transformations); this is the very definition of geometry.** According to the *Erlangen Program* proposed by Felix Klein, geometry is the study of properties that remain invariant under a certain group of transformations. If the transformations are restricted to rigid transformations (translation, rotation, reflection), it is Euclidean geometry; if the transformations are general linear transformations, it is affine geometry. Riemannian geometry, however, is concerned with objective entities that are independent of *any* coordinate system. For example, if I have a vector with a fixed direction and magnitude, it might be $(1, 1)$ in a Cartesian coordinate system and $(\sqrt{2}, \pi/4)$ in a polar coordinate system. Although the components differ between the two coordinate systems, they both refer to the same vector. That is to say, the vector itself is an objectively existing entity, independent of the coordinates used. From an algebraic perspective, as long as they can be transformed into one another through some coordinate transformation, we consider them to be the same thing.

Therefore, when learning Riemannian geometry, it is always beneficial to think in the direction of “objective entities.”

[Click to view original SVG: Geodesics on a plane](#)

Figure 1: Geodesics on a plane (Placeholder for assets/3977/426850218.svg)

With a metric, one can naturally introduce the entity known as a “geodesic.” From a narrow perspective, it is the shortest line between two points—a generalization of the concept of a straight line segment in flat space. (Actual geodesics are not necessarily the shortest, but let us not dwell on the details for now; this does not hinder our understanding, as a geodesic is at least locally shortest). It is not difficult to imagine that once two points are determined, the shortest line between them is fixed regardless of the coordinates used; thus, this is clearly an objective entity. A simple analogy is that no matter how you transform the coordinates, the extreme points of the graph of a function $f(x)$ are always fixed—whether you change the coordinates or not, they are there, exactly where they should be.

[SVG: Geodesics on a sphere](#)

Figure 2: Geodesics on a sphere

[SVG: Geodesics on an isothermal surface](#)

Figure 3: Geodesics on an isothermal surface

Mathematically, the distance between two points $\mathbf{x}^1$ and $\mathbf{x}^2$ is naturally:

$$s = \int_{\mathbf{x}^1}^{\mathbf{x}^2} ds = \int_{\mathbf{x}^1}^{\mathbf{x}^2} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \quad (16)$$

Therefore, finding a geodesic means finding the function among all functions passing through $\mathbf{x}^1$ and $\mathbf{x}^2$ that minimizes the above integral. This is a problem of the calculus of variations. Unfortunately, many mathematics students have not studied the calculus of variations, but I will still use this approach because it is a very natural line of thought. As we will see later, it also provides a simplified scheme for calculating connections.

The idea of variation is actually very simple, much like finding a derivative, with the difference being an additional step of integration by parts. To find the extremum of a function, we take the derivative and set it to zero; for the extremum of a functional, we take the variation of the functional and set it to zero.

$$\begin{aligned} \delta s &= \int_{\mathbf{x}^1}^{\mathbf{x}^2} \delta \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \\ &= \int_{\mathbf{x}^1}^{\mathbf{x}^2} \frac{\delta(g_{\mu\nu} dx^\mu dx^\nu)}{2ds} \\ &= \int_{\mathbf{x}^1}^{\mathbf{x}^2} \left(\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} dx^\nu \delta x^\alpha + g_{\mu\nu} \frac{dx^\mu}{ds} d\delta x^\nu \right) \\ &= \int_{\mathbf{x}^1}^{\mathbf{x}^2} \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} dx^\nu \delta x^\alpha + g_{\mu\nu} \frac{dx^\mu}{ds} \delta x^\nu \Big|_{\mathbf{x}^1}^{\mathbf{x}^2} - \int_{\mathbf{x}^1}^{\mathbf{x}^2} d \left(g_{\mu\nu} \frac{dx^\mu}{ds} \right) \delta x^\nu \\ &= \int_{\mathbf{x}^1}^{\mathbf{x}^2} \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} dx^\nu \delta x^\alpha - \int_{\mathbf{x}^1}^{\mathbf{x}^2} d \left(g_{\mu\nu} \frac{dx^\mu}{ds} \right) \delta x^\nu \\ &= \int_{\mathbf{x}^1}^{\mathbf{x}^2} \left[\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} dx^\nu \delta x^\alpha - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} dx^\alpha \delta x^\nu - g_{\mu\nu} d \left(\frac{dx^\mu}{ds} \right) \delta x^\nu \right] \\ &= \int_{\mathbf{x}^1}^{\mathbf{x}^2} \left[\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} dx^\nu - \frac{\partial g_{\mu\alpha}}{\partial x^\nu} \frac{dx^\mu}{ds} dx^\nu - g_{\mu\alpha} d \left(\frac{dx^\mu}{ds} \right) \right] \delta x^\alpha \end{aligned} \quad (17)$$

The term resulting from integration by parts vanishes because we specified that we are looking for the function among all those passing through $\mathbf{x}^1$ and $\mathbf{x}^2$, so at the boundaries we have $\delta x^\nu(\mathbf{x}^1) = \delta x^\nu(\mathbf{x}^2) = 0$. Finally, since δx^α is arbitrary, for $\delta s = 0$ to hold, it must be that:

$$\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} dx^\nu - \frac{\partial g_{\mu\alpha}}{\partial x^\nu} \frac{dx^\mu}{ds} dx^\nu - g_{\mu\alpha} d \left(\frac{dx^\mu}{ds} \right) = 0 \quad (18)$$

With a little rearrangement, we obtain:

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \quad (19)$$

where

$$\Gamma_{\alpha\beta}^\mu = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right) \quad (20)$$

is called the Christoffel symbol (of the second kind), also known as the connection coefficient. We will soon understand the meaning of this term. Here $g^{\mu\nu}$ is the inverse matrix of $g_{\mu\nu}$, i.e.,

$$g^{\mu\alpha} g_{\alpha\nu} = \delta_\nu^\mu \quad (21)$$

Furthermore, the variation of s is equivalent to the variation of the following function S using s directly as a parameter:

$$S = \frac{1}{2} \int g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} ds \quad (22)$$

Since S does not contain a square root and has a simpler form, it can be substituted directly into the Euler-Lagrange equations for calculation. This is sometimes more convenient than directly varying the original s .

A Powerful Calculation Tool

Equation (20) provides the method for calculating the connection coefficients $\Gamma_{\alpha\beta}^{\mu}$. It involves partial derivatives, inverse matrix calculations, and index summations, making it a very complex term. Readers can try to calculate it to experience the difficulty involved. However, sometimes after very complex calculations, we find that many terms of $\Gamma_{\alpha\beta}^{\mu}$ are zero. That is to say, the calculation process is complex, but the result is simple. This motivates us to seek a simplification technique.

In fact, we derived the geodesic equation from a variational approach, and this approach itself is a powerful tool for calculating $\Gamma_{\alpha\beta}^{\mu}$. Chapter 14, “Calculation of Curvature,” in the famous gravitation “bible” *Gravitation* (MTW) discusses related topics. (This is provided the result is simple; if the result itself is complex, then there are no simplification tricks). For example, consider the case of spherical coordinates:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad x^1 = r, x^2 = \theta, x^3 = \phi \quad (23)$$

This is equivalent to the variation:

$$s = \int \frac{1}{2} \left[\left(\frac{dr}{ds} \right)^2 + r^2 \left(\frac{d\theta}{ds} \right)^2 + r^2 \sin^2 \theta \left(\frac{d\phi}{ds} \right)^2 \right] ds \quad (24)$$

Using the Euler-Lagrange equations, we can quickly write:

$$\begin{cases} \frac{d^2 r}{ds^2} = r \left(\frac{d\theta}{ds} \right)^2 + r \sin^2 \theta \left(\frac{d\phi}{ds} \right)^2 \\ \frac{d}{ds} \left(r^2 \frac{d\theta}{ds} \right) = r^2 \sin \theta \cos \theta \left(\frac{d\phi}{ds} \right)^2 \\ \frac{d}{ds} \left(r^2 \sin^2 \theta \frac{d\phi}{ds} \right) = 0 \end{cases} \quad (25)$$

Rearranging gives:

$$\begin{cases} \frac{d^2 r}{ds^2} = r \left(\frac{d\theta}{ds} \right)^2 + r \sin^2 \theta \left(\frac{d\phi}{ds} \right)^2 \\ \frac{d^2 \theta}{ds^2} = -\frac{2}{r} \frac{dr}{ds} \frac{d\theta}{ds} + \sin \theta \cos \theta \left(\frac{d\phi}{ds} \right)^2 \\ \frac{d^2 \phi}{ds^2} = -\frac{2}{r} \frac{dr}{ds} \frac{d\phi}{ds} - \frac{2 \cos \theta}{\sin \theta} \frac{d\theta}{ds} \frac{d\phi}{ds} \end{cases} \quad (26)$$

Comparing this with the geodesic equation (18), we obtain:

$$\begin{aligned} \Gamma_{22}^1 &= -r, & \Gamma_{33}^1 &= -r \sin^2 \theta \\ \Gamma_{12}^2 &= \Gamma_{21}^2 = \frac{1}{r}, & \Gamma_{33}^2 &= -\sin \theta \cos \theta \\ \Gamma_{13}^3 &= \Gamma_{31}^3 = \frac{1}{r}, & \Gamma_{23}^3 &= \Gamma_{32}^3 = \frac{\cos \theta}{\sin \theta} \end{aligned} \quad (27)$$

All others are zero. As can be seen, if one is proficient in the calculus of variations (which does not require much effort), it can help us quickly find the connection coefficients without getting bogged down in various index summations.

Of course, in the computer age, few people calculate the various connection coefficients of complex metrics by hand. However, for certain simple metrics, calculating them personally allows us to gain a deeper understanding of them.

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